1. Answer
(i) No.

Let $\varphi : \mathbb{Z}/5\mathbb{Z} \mapsto S_4$ be a homomorphism and $e \neq x \in \mathbb{Z}/5\mathbb{Z}$. Then $o(\varphi(x))|o(x) = 5$ and $o(\varphi(x))|o(S_4) = 24$. So that $o(\varphi(x)) = 1$, $\varphi(x) = e'$ identinty in S_4. Therefore φ is trivial homomorphism.
(ii) Yes.

Let $o(G) = 9 = 3^2$. We know that $o(G) = o(Z(G)) + \sum_{a \notin Z(G)} [G, C(a)]$, so $o(Z(G)) \neq 1$. Therefore $o(Z(G)) = 3$ or 9. If $o(Z(G)) = 9$, $Z(G) = G$, implies G is abelian. Suppose $o(Z(G)) = 3$. Then we have $o(G/Z(G)) = 3$, so $G/Z(G)$ is cyclic. Hence G is abelian.
(iii) Yes.

Recall that centre of a group G, $Z(G) = \{x \in G | gx = xg,\ \forall g \in G\}$. To show $Z(S_n) = \{e\}$, it is enough to show that for all $\sigma \in S_n,\ e \neq \sigma$ there exists $\tau \in S_n$ such that $\sigma\tau \neq \tau\sigma$. Since $\sigma \neq e$, we have $\sigma(a) = b$ for some $a \neq b,\ a, b \in \{1, 2, \ldots, n\}$. As $n \geq 3$, we can choose $c \in \{1, 2, \ldots, n\}$, such that $c \neq b\ (\neq a)$. Now take $\tau = (b\ c)$. Then $\sigma\tau(a) = \sigma(a) = b$ and $\tau\sigma(a) = \tau(b) = c$. Therefore $\sigma\tau \neq \tau\sigma$.

(iv) No.

$\{(1),\ (12)(34),\ (13)(24),\ (14)(23)\}$ is a normal subgroup of A_4.
2. Answer
(a) $Q_8 = \{\pm 1,\ \pm i,\ \pm j,\ \pm k\}$ is a non abelian group under multiplication. $\{1,\ -1\}$, $\{\pm 1,\ \pm i\}$, $\{\pm 1,\ \pm j\}$, $\{\pm 1,\ \pm k\}$ are the sub groups of Q_8 and these are all normal subgroups.
(b) Define $\varphi : \mathbb{R}/\mathbb{Z} \mapsto S^1 = \{z \in C : |z| = 1\}$ by $\varphi(x + \mathbb{Z}) = e^{2\pi i x},\ 0 \leq x < 1$. Then, clearly φ is an isomorphism.
(c) If φ is from $\mathbb{Q}/\mathbb{Z}$ to S^1. Then $\varphi(p/q + \mathbb{Z}) = e^{2\pi i p/q},\ 0 \leq p/q < 1$. Therefore $\varphi(\mathbb{Q}/\mathbb{Z}) = \{z \in \mathbb{C} : z^n = 1,\ n$ th roots of unity, for some $n \in \mathbb{N}\}$.
3. (a) For any $\sigma \in S_n$ and any $d \leq n$, we have

$$\sigma(12 \ldots d)\sigma^{-1} = (\sigma(1)\sigma(2) \ldots \sigma(d)).$$

So that any conjugate of a d-cycle is again a d-cycle. We know that every permutation is a product of disjoint cycles, so the cycle types of conjugate permutations are same.

Conversly suppose $\varphi,\ \psi$ are two permutations of same cycle type.
Let

$$\varphi = (a_1 \ldots a_r)(a_{r+1 \cdots a_s}) \cdots (a_l \cdots a_m)$$
$$\psi = (b_1 \ldots b_r)(b_{r+1 \cdots b_s}) \cdots (b_l \cdots b_m).$$

Now define $\sigma \in S_n$ by $\sigma(a_i) = b_i,\ 1 \leq i \leq m$. Then

$$\begin{aligned}
\sigma\varphi\sigma^{-1} &= \sigma(a_1 \ldots a_r)(a_{r+1 \cdots a_s}) \cdots (a_l \cdots a_m)\sigma^{-1} \\
&= \sigma(a_1 \ldots a_r)\sigma^{-1}\sigma(a_{r+1 \cdots a_s})\sigma^{-1} \cdots \sigma(a_l \cdots a_m)\sigma^{-1} \\
&= (\sigma(a_1) \ldots \sigma(a_r))(\sigma(a_{r+1}) \cdots \sigma(a_s)) \cdots (\sigma(a_l) \cdots \sigma(a_m)) \\
&= (b_1 \ldots b_r)(b_{r+1 \cdots b_s}) \cdots (b_l \cdots b_m) \\
&= \psi.
\end{aligned}$$

Therefore $\varphi,\ \psi$ are conjugates.

(b) We know that $|[S_7; C_{S_7}(\sigma)]| = |cl_{S_7}(\sigma)|$ where $cl_{S_7}(\sigma)$ is the conjugancy class of σ.

Now we have

$$|C_{S_7}(\sigma)| = \frac{|S_7|}{|[S_7; C_{S_7}(\sigma)]|} = \frac{|S_7|}{|cl_{S_7}(\sigma)|}.$$

But

$$|cl_{S_7}(\sigma)| = \frac{7!}{[(3)^1 1!][(1)^4 4!]} = \frac{7!}{3 \times 4!}.$$

Therefore $|C_{S_7}(\sigma)| = 3 \times 4! = 72$.

For any $\alpha \in S_7$, we have $\alpha\sigma\alpha^{-1} = (\alpha(2)\ \alpha(4)\ \alpha(6))$. So

$$\alpha\sigma\alpha^{-1} = \sigma \Leftrightarrow (\alpha(2)\ \alpha(4)\ \alpha(6)) = (2\ 4\ 6).$$

Therefore $(\alpha(2)\,\alpha(4)\,\alpha(6)) \in \{(2\,4\,6),\ (6\,2\,4),\ (4\,6\,2)\} = A$ (say). Hence $C_{S_7}(\sigma) = \{\alpha \in S_7 | (\alpha(2)\,\alpha(4)\,\alpha(6)) \in A\}$ and it has 72 elements.

4.(a) See Thereom 9.4 (page 179), 'Contemporary Abstract Algebra (Book)' by Joseph A. Gallian.

(b) Let $G = \langle x \rangle$. For $d,\ 1 \leq d \leq n$ and $(d,\ n) = 1$, we define $\phi_d : G \mapsto G$ by $\phi_d(x^i) = x^{di}$. Then $\phi_d(x^i x^j) = \phi_d(x^i)\phi_d(x^j)$, so it is a homomorphism. If $\phi_d(x^i) = \phi_d(x^j)$, then $x^{d(i-j)} = e$. So $n|d(i-j)$. As $(d,\ n) = 1$ and $1 \leq i, j \leq n$, we have $i = j$. Therefore ϕ_d is one-one. Hence ϕ_d is an automorphism.

Let f be any automorphism of G. Then $f(x) \in G$. So we have $f(x) = x^m$ for some $1 \leq m \leq n$. Since f is an automorphism, $o(f(x)) = o(x) = n$. Which implies $(m,\ n) = 1$. Therefore $f = \phi_d$, for some $d,\ (d,\ n) = 1$. So that ϕ_d are all only the automorphisms of G. The set of automorphisms $\{\phi_d : (d,\ n) = 1\}$ forms a group and order of this group is $\phi(n)$.

5. (a) See Application 1 (page 61), 'Topics in Algebra (Book)' by i.n. herstein.

(b) See Thereom 2.11.3 (page 87), 'Topics in Algebra (Book)' by i.n. herstein.